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Improved Determination of the b Quark Mass from Spectroscopy

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Abstract. Using recently evaluated contributions (including a novel one calculated here), we present updated values for the pole mass and $\overline{MS}$ mass of the b quark: $m_b = 5022 \pm 58$ MeV, for the pole mass, and $\bar{m}_b(\bar{m}_b) = 4286 \pm 36$ MeV for the $\overline{MS}$ one. These values are accurate including, respectively, $O(\alpha_s^5 \log \alpha_s)$ and $O(\alpha_s^3)$ corrections and, in both cases, leading orders in the ratio m_c^2/m_b^2.

One of the sources of information for the quark masses is quarkonium spectroscopy. By evaluating the $\bar{b}b$ potential including relativistic and radiative corrections, as well as leading nonperturbative effects,[1,2,3] and using this in a perturbative expansion, it has been possible to find values of the *pole* quark masses with increasing accuracy;[2,3,4] in this note we will go up to fourth and leading fifth order, in the approximation of neglecting "light" (u, d, s) quark masses and to leading order (actually, $O(\alpha_s m_c^2/m_b^2)$) in the c quark mass. The connection with the $\overline{\rm MS}$ mass has been known for some time to one and two loops[5]: very recently, a three loop evaluation has been completed. Coupling this with the pole mass evaluations, we now have an order α_s^3 result for the $\overline{\rm MS}$ mass. We review here briefly this.

1. *$m_b - \bar{m}_b(\bar{m}_b)$ connection*

Write, for a heavy quark,

$$\bar{m}(\bar{m}) \equiv m/\{1+\delta_1+\delta_2+\delta_3+\cdots\}; \tag{1a}$$

m here denotes the *pole* mass, and $\bar{m}$ is the $\overline{\rm MS}$ one. One has

$$\delta_1 = C_F \frac{\alpha_s(\bar{m})}{\pi}, \quad \delta_2 = c_2 \left(\frac{\alpha_s(\bar{m})}{\pi}\right)^2, \quad \delta_3 = c_3 \left(\frac{\alpha_s(\bar{m})}{\pi}\right)^3. \tag{1b}$$

Here α_s is to be calculated to three loops:

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 L}\left\{1 - \frac{\beta_1 \log L}{\beta_0^2 L} + \frac{\beta_1^2 \log^2 L - \beta_1^2 \log L + \beta_2\beta_0 - \beta_1^2}{\beta_0^4 L^2}\right\}$$

with

$$L = \log\frac{\mu^2}{\Lambda^2}; \quad \beta_0 = 11 - \tfrac{2}{3}n_f, \quad \beta_1 = 102 - \tfrac{38}{3}n_f, \quad \beta_2 = \tfrac{2857}{2} - \tfrac{5033}{18}n_f + \tfrac{325}{54}n_f^2.$$

The coefficient c_2 has been evaluated by Gray et al.[5], and reads

$$c_2 = -K + 2C_F, \tag{1c}$$

$$\begin{aligned} K = {} & K_0 + \sum_{i=1}^{n_f} \Delta\left(\frac{m_i}{m}\right), \quad K_0 = \tfrac{1}{9}\pi^2 \log 2 + \tfrac{7}{18}\pi^2 - \tfrac{1}{6}\zeta(3) + \tfrac{3673}{288} - \left(\tfrac{1}{18}\pi^2 + \tfrac{71}{144}\right)(n_f+1) \\ & \simeq 16.11 - 1.04\, n_f; \quad \Delta(\rho) = \tfrac{4}{3}\left[\tfrac{1}{8}\pi^2\rho - \tfrac{3}{4}\rho^2 + \cdots\right]. \end{aligned} \tag{1d}$$

m_i are the (pole) masses of the quarks strictly lighter than m, and n_f is the number of these. For the b quark case, $n_f = 4$ and only the c quark mass has to be considered; we will take $m_c = 1.8$ GeV (see Table 1 below) for the calculations.

The coefficient c_3 was recently calculated by Melnikov and van Ritbergen,[6] where the exact expression may be found. Neglecting now the m_i,

$$c_3 \simeq 190.389 - 26.6551 n_f + 0.652694 n_f^2. \tag{1e}$$

For the b, c quarks, with α_s as given below,

$$\begin{aligned} \delta_1(b) =&0.090, \quad \delta_1(c) =0.137,\\ \delta_2(b) =&0.045, \quad \delta_2(c) =0.108,\\ \delta_3(b) =&0.029; \quad \delta_3(c) =0.125. \end{aligned} \tag{2}$$

From these values we conclude that, for the c quark, the series has started to diverge at second order, and it certainly diverges at order α_s^3. For the b quark the series is at the edge of convergence for the α_s^3 contribution.

Take now as input parameters

$$\Lambda(n_f = 4, \text{ three loops}) = 0.283 \pm 0.035 \ \text{GeV} \ \left[\, \alpha_s(M_Z^2) \simeq 0.117 \pm 0.024 \,\right]$$

(ref. 7) and for the gluon condensate, very poorly known, the value $\langle \alpha_s G^2 \rangle = 0.06 \pm 0.02 \ \text{GeV}^4$. From the mass of the Υ particle we have a very precise determination for the pole mass of the b quark. This determination is correct to order α_s^4 and including leading $O(m_c^2/m_b^2)$ and leading nonperturbative corrections as well as the α_s^5 corrections proportional to $\log \alpha_s$; the details of it will be given below. With the renormalization point $\mu = m_b C_F \alpha_s$ we have,

$$\begin{aligned} m_b =&5022 \pm 43\,(\Lambda) \ \mp 5\,(\langle \alpha_s G^2 \rangle)^{-31}_{+37} \,(\text{vary } \mu^2 \text{ by } 25\%) \ \pm 38\,(\text{other th. uncert.})\\ =&5022 \pm 58 \ \text{MeV}\,. \end{aligned} \tag{3a}$$

Here we append (Λ) to the error induced by that of Λ, and likewise $(\langle \alpha_s G^2 \rangle)$ tags the error due to that of the condensate. The error labeled (other th. uncert.) includes also the error evaluated in ref. 8; the rest is as in ref. 3.

Using the three loop relation (1) of the pole mass to the $\overline{\text{MS}}$ mass we then find

$$\bar{m}_b(\bar{m}_b) = 4284 \pm 7 \ (\Lambda) \mp 5 \ (\langle \alpha_s G^2 \rangle) \pm 35 \,(\text{other th. uncert.}) = 4284 \pm 36 \ \text{MeV}\,. \tag{3b}$$

The slight dependence of $\bar{m}$ on Λ when evaluated in this way was already noted in ref. 2.

There is another way of obtaining $\bar{m}$, which is to express directly the mass of the Υ in terms of it, using Eq. (1) and the order α_s^3 formula for the Υ mass in terms of the pole mass (see e.g. ref. 2). One finds, for $n_f = 4$, and neglecting m_c^2/m_b^2,

$$M(\Upsilon) = 2\bar{m}(\bar{m}) \left\{ 1 + C_F \frac{\alpha_s(\bar{m})}{\pi} + 7.559 \left(\frac{\alpha_s(\bar{m})}{\pi} \right)^2 + [66.769 + 18.277\,(\log C_F + \log \alpha_s(\bar{m}))] \left(\frac{\alpha_s(\bar{m})}{\pi} \right)^3 \right\}. \tag{4a}$$

(One could add the leading nonperturbative contributions to (4a) à la Leutwyler–Voloshin in the standard way; see e.g. refs. 2, 3, 9). This method has been at times advertised as improving the convergence, allegedly because the $\overline{\text{MS}}$ mass does not suffer from nearby renormalon singularities. But a close look to (4a) does not seem to bear this out. To an acceptable $O(\alpha_s^4)$ error we can replace $\log(\alpha_s(\bar{m}))$ by $\log(\alpha_s(M(\Upsilon/2))$ above. With Λ as before (4a) then becomes

$$M(\Upsilon) = 2\bar{m}_b(\bar{m}_b) \left\{ 1 + C_F \frac{\alpha_s(\bar{m})}{\pi} + 7.559 \left(\frac{\alpha_s(\bar{m})}{\pi} \right)^2 + 43.502 \left(\frac{\alpha_s}{\pi} \right)^3 \right\}. \tag{4b}$$

This does not look particularly convergent, and is certainly not an improvement over the expression using the pole mass, where one has for the choice[3] $\mu = C_F m_b \alpha_s$, and still neglecting the masses of quarks lighter than the b,

$$M(\Upsilon) = 2m_b \left\{ 1 - 2.193 \left(\frac{\alpha_s(\mu)}{\pi} \right)^2 - 24.725 \left(\frac{\alpha_s(\mu)}{\pi} \right)^3 - 458.28 \left(\frac{\alpha_s(\mu)}{\pi} \right)^4 + 897.93\,[\log \alpha_s] \left(\frac{\alpha_s}{\pi} \right)^5 \right\}. \tag{5}$$

To order three, (5) is actually better[1] than (4b). What is more, logarithmic terms appear in (4) at order α_s^3, while for the pole mass expression they first show up at α_s^5. Finally, the direct formula for $M(\Upsilon)$ in terms of the $\overline{\rm MS}$ mass presents the extra difficulty that the *nonperturbative* contribution becomes larger than than what one has for the expression in terms of the pole mass (~ 80 against ~ 9 MeV), because of the definition of the renormalization point. With the purely perturbative expression (4) plus leading nonperturbative (gluon condensate) correction one finds the value $\bar{m}_b(\bar{m}_b) = 4167\,\text{MeV}$, rather low.

2. *Improved determination of* m_b

Eq. (5) was deduced neglecting the masses of all quarks lighter than the b. The influence of the nonzero mass of the c quark, the only worth considering, will be evaluated now. To leading order it only contributes to the $\bar{b}b$ potential through a c-quark loop in the gluon exchange diagram (diagram f_2 in ref. 2). The momentum space potential generated by a nonzero mass quark through this mechanism is then, in the nonrelativistic limit,

$$\widetilde{V}_{c\,\text{mass}} = -\frac{8C_F T_F \alpha_s^2}{\mathbf{k}^2} \int_0^1 \mathrm{d}x\, x(1-x) \log \frac{m_c^2 + x(1-x)\mathbf{k}^2}{\mu^2}. \tag{6}$$

We expand in powers of $m_c^2/\mathbf{k}^2$. The zeroth term is already included in (5). The first order correction is

$$\delta_{c\,\text{mass}} \widetilde{V} = -\frac{8C_F T_F \alpha_s^2 m_c^2}{\mathbf{k}^4}. \tag{7}$$

In x-space,

$$\delta_{c\,\text{mass}} V = \frac{C_F T_F \alpha_s^2 m_c^2}{\pi}\, r. \tag{8}$$

This induces the shift in the mass of the Υ of

$$\delta_{c\,\text{mass}} M(\Upsilon) = \frac{3T_F \alpha_s}{\pi} \frac{m_c^2}{m_b^2}\, m_b,$$

so Eq. (5) is modified to

$$\begin{aligned} M(\Upsilon) =& 2m_b \Bigg\{ 1 - 2.193 \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 - 24.725 \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 - 458.28 \left(\frac{\alpha_s(\mu)}{\pi}\right)^4 \\ & +897.93\, [\log \alpha_s] \left(\frac{\alpha_s}{\pi}\right)^5 + \frac{3T_F \alpha_s}{2\pi} \frac{m_c^2}{m_b^2} \Bigg\}. \end{aligned} \tag{8}$$

This produces the value quoted in (3a). Note that the (new) correction of order m_c^2/m_b^2 is responsible for a shift in m_b of

$$\delta_{c\,\text{mass}} m_b = -35 \ \text{MeV},$$

substantially larger than the $\alpha_s^5 \log \alpha_s$ correction evaluated by Brambilla et al.[4] which, for the renormalization point $\mu = m_b C_F \alpha_s$, gives

$$\delta_{[\alpha_s^5 \log \alpha_s]} m_b = \tfrac{1}{2} m_b [C_F + \tfrac{3}{2} C_A] C_F^4 \alpha_s^5 (\log \alpha_s)/\pi \simeq -8\,\text{MeV}\,.$$

We collect in the table the determinations of the b quark mass based on spectroscopy, to increasing accuracy. The *stability* of the numerical values of the pole mass is remarkable: the pole masses all lie within each other error bars. The $\overline{\rm MS}$ ones show more spread.

[1] The convergence of Eq. (5) is still improved if one solves exactly the purely coulombic part of the static potential, as was done in refs. 2, 3, where we send for details. For example, the $O(\alpha_s^4)$ term becomes $-232.12(\alpha_s/\pi)^4$. This is the method we used to get the values of m_b here.

Reference	m_b(pole)	$\bar{m}_b(\bar{m}_b^2)$	m_c(pole)	$\bar{m}_c(\bar{m}_c^2)$
TY	4971 ± 72	4401^{+21}_{-35}	$1585 \pm 20\,(^*)$	$1321 \pm 30\,(^*)$
PY	5065 ± 60	4455^{+45}_{-29}	1866^{+215}_{-133}	1542^{+163}_{-104}
Here	5022 ± 58	4286 ± 36	–	–

Table 1. b and c quark masses. $(^*)$ Systematic errors not included.

TY: Titard and Ynduráin[2]. $O(\alpha_s^3)$ plus $O(\alpha_s^3)v$, $O(v^2)$ for m; $O(\alpha_s^2)$ for $\bar{m}$. Rescaled for $\Lambda(n_f = 4) = 283$ MeV.
PY: Pineda and Ynduráin[3]. Full $O(\alpha_s^4)$ for m; $O(\alpha_s^2)$ for $\bar{m}$. Rescaled for $\Lambda(n_f = 4) = 283$ MeV.
Here: This calculation. $O(\alpha_s^4)$, $O(\alpha_s m_c^2/m_b^2)$ and $O(\alpha_s^5 \log \alpha_s)$ for m; $O(\alpha_s^3)$ and $O(\alpha_s^2 m_c^2/m_b^2)$ for $\bar{m}$. Values not given for the c quark, as the higher order terms are as large as the leading ones.

We finally remark that the values of m_b quoted e.g. in the Table 1 were *not* obtained solving Eq. (8), but solving exactly the coulombic part of the interaction, and perturbing the result (see refs. 2, 3 for details). We also note that, in the determinations of m_b, the new pieces, $O(\alpha_s m_c^2/m_b^2)$ and $O(\alpha_s^5 \log \alpha_s)$, have been evaluated to first order; in particular, we have included the corresponding shifts in the *central* values, not in the errors. If we included these, the errors would decrease by some 7%.

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